MAE106 Mechanical Systems Laboratory: Time and Frequency Domain Notes

1. Why do engineers analyze systems in both the time and frequency domain?

Why the time domain? We live in the time domain.

Typical questions:
- How does the system respond to a step input? Example: 0-60 mph
- How does the system respond to a impulse input? Example: bump suspension
- How fast does the system respond? (Useful #: time constant)
- Does it overshoot ?
- Does it oscillate ?

Input → $\boxed{\text{System}}$ → output

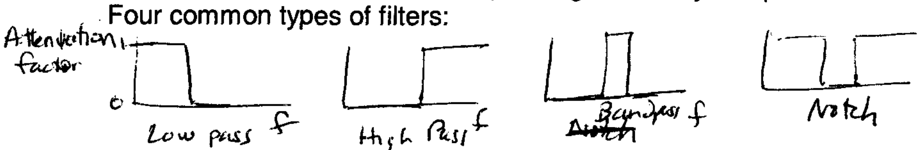
attenuated

Why the frequency domain?

a. Intuition

Systems act like filters, responding differently to inputs at different frequencies

Four common types of filters:



b. Ease – sometimes its easier to solve differential equations in the frequency domain (Laplace Transform)

2. What is a transfer function and what is a frequency response?

A linear differential equation in the time domain becomes a transfer function in the freq. domain.

To see this take the Laplace transform of a differential equation:

$$\frac{dx}{dt} + ax = u$$

Assumes initial conditions are zero

Recall $\mathcal{L}\left(\frac{dx}{dt}\right) = s\,\mathcal{L}(x(t))$

$\mathcal{L}(x_1+x_2) = \mathcal{L}(x_1) + \mathcal{L}(x_2)$

$$sX(s) + aX(s) = U(s) \quad X(s) = \frac{1}{s+a}U(s) = H(s)\,U(s)$$

FACT: The transfer function tells how a system responds to any input in the frequency domain. The output is just the input multiplied by the transfer function.

$U(s) \rightarrow \boxed{H(s)} \rightarrow X(s) = H(s)U(s)$

The transfer function also tells how a system responds to a sinusoidal input.

FACT: Using Laplace Transforms, it is possible to prove that: sine wave in ⇒ sine wave out scaled ~~shifted~~

The transfer function tells how much an input sine wave is scaled and shifted as a function of its frequency.

$\sin(\omega t) \rightarrow \boxed{G(s)} \rightarrow |G(j\omega)|\sin(\omega t + \phi_{G(j\omega)})$

Note $G(j\omega)$ is a complex variable, so it has a magnitude + phase

Knowing these two things means you know the frequency response of the system, which is characterized by the Magnitude and the phase responses

These facts are very useful when combined with two other facts:

FACT: Any signal can be represented as the sum of sinusoids. Fourier analysis

FACT: The response of a linear system to the sum of two inputs is the sum of the individual outputs.

<u>THESE FACTS LET US THINK OF LINEAR SYSTEMS AS FILTERS.</u>

$x_1 \rightarrow y_1$

$x_2 \rightarrow y_2$

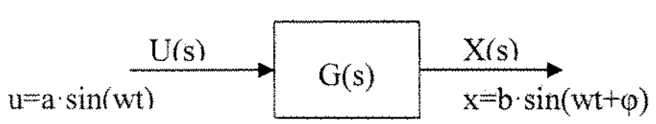
$x_1 + x_2 \rightarrow y_1 + y_2$

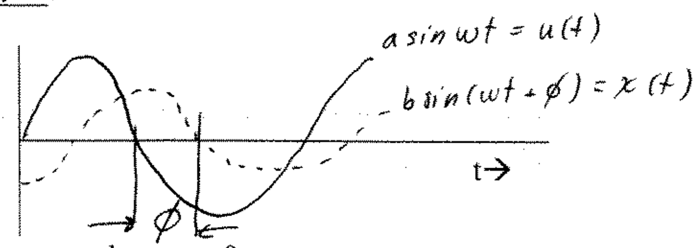
Linear System

any signal $\rightarrow \sum$ sines $\rightarrow \boxed{G(s)} \rightarrow \sum$ scaled shifted sines $\rightarrow$ output signal

Mechanical Systems Laboratory
Frequency Response: Sine wave in, sine wave out with different amplitude and phase

Overview:
If we input a sine wave to a stable linear system, the output will be a _sine_ wave of the same frequency with different _amplitude_ and _phase_.





For the signals shown x(t) _lags_ u(t) by φ°, and this means that φ $<$ 0.

Mathematical model of the system:
How can we predict the output amplitude and phase? Consider a general n[th] order linear system:

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_0 x = b_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + b_{n-2}\frac{d^{n-2}x}{dt^{n-2}} + \ldots + b_0 u$$

If we take the Laplace Transform:

$$X(\underbrace{s^n + a_{n-1}s^{n-1} + \ldots + a_0}_{A(s)}) = U(\underbrace{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}_{B(s)}) + IC(s) \;\rightarrow\; X = \frac{B(s)}{A(s)}U + \frac{IC(s)}{A(s)}$$

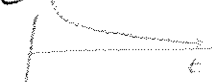
A(s) is called the _characteristic polynomial_ of the system, and we can write it in factored form:

$$A(s) = (s-s_1)(s-s_2)\ldots(s-s_n)$$

What happen if U(s)=0?

fractional expansion

$$X = \frac{IC(s)}{A(s)} \;\rightarrow\; X = \frac{c_1}{s-s_1} + \frac{c_2}{s-s_2} + \ldots + \frac{c_n}{s-s_n} \;\xrightarrow{\mathcal{L}^{-1}}\; x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots + c_n e^{s_n t}$$



The system is stable if $\mathrm{Re}(s_i) < 0$, where s_i are the _roots_ of the _characteristic polynomial_ A(s).
Assume we have a stable system, and we apply a sine wave to the system:

$$u(t) = a \cdot \sin wt \quad \mathcal{L}\rightarrow \quad U(s) = \frac{a w}{s^2 + w^2}$$

$$X(s) = \frac{B(s)}{A(s)}\frac{a w}{s^2+w^2} + \frac{IC(s)}{A(s)}$$

since $\mathrm{Re}(s_i) < 0 \rightarrow$ decays to zero

Since A(s) is stable, the contribution of the response due to the initial conditions decays to zero. Use partial fraction expansion to find the inverse Laplace transform.

$$X(s) = \frac{B(s)}{A(s)}\frac{a w}{s^2+w^2} = \underbrace{\frac{k_1}{s+jw} + \frac{k_2}{s-jw}}_{\text{roots of } s^2+w^2} + \underbrace{\frac{k_3}{s-s_1} + \frac{k_4}{s-s_2} + \ldots}_{\text{roots of } A(s)}$$

The inverse Laplace transform has the form: (Trick: $L\left[e^{-at}\right] = \frac{1}{s+a}$)

$$x(t) = k_1 e^{-jwt} + k_2 e^{jwt} + k_3 e^{s_1 t} + k_4 e^{s_2 t} + \ldots$$

0 0

because $\mathrm{Re}(s_i) < 0$

The terms $e^{s_i t}$ decay to zero, because $\text{Re}(s_i) < 0$. Use partial fraction expansion to find K_1 and K_2.

$$(s+j\omega)\frac{B(s)}{A(s)}\frac{a\omega}{s^2+\omega^2} = \frac{k_1(s+j\omega)}{s+j\omega} + \frac{k_2(s+j\omega)}{s-j\omega}\Bigg|_{s=-j\omega} \Rightarrow k_1 = \frac{B(s)}{B(s)}\frac{a\omega}{s^2+\omega^2}(s+j\omega)\Bigg|_{s=-j\omega} \Rightarrow$$

$$k_1 = \frac{B(s)}{A(s)}\frac{a\omega\,\cancel{(s+j\omega)}}{\cancel{(s+j\omega)}(s-j\omega)}\Bigg|_{s=-j\omega} = \frac{B(-j\omega)}{A(-j\omega)}\frac{a\omega}{(-2j\omega)} = -\frac{a}{2j}G(-j\omega)$$

Similarly, $k_2 = \frac{a}{2j}G(j\omega)$

We can write $G(jw)$ in polar coordinates as: $G(jw) = |G(jw)|e^{j\varphi}$, where $\varphi = \angle G(jw)$. So that we can express $G(-jw)$ as: (hint: $G(-jw)$ is the complex conjugate of $G(jw)$). $G(-j\omega) = |G(j\omega)|e^{-j\varphi}$

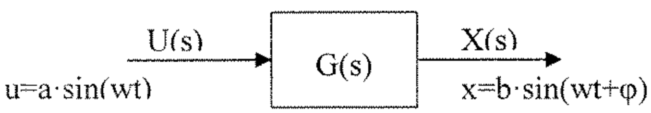
$$x(t) = k_1 e^{-j\omega t} + k_2 e^{j\omega t} = -\frac{a}{2j}G(-j\omega)e^{-j\omega t} + \frac{a}{2j}G(j\omega)e^{j\omega t} =$$

$$= -\frac{a}{2j}|G(j\omega)|e^{-j\phi}e^{-j\omega t} + \frac{a}{2j}|G(j\omega)|e^{j\varphi}e^{j\omega t} = \frac{a}{2j}|G(j\omega)|\left(e^{j(\omega t+\phi)t} - e^{-j(\omega t+\phi)}\right)$$

But using Euler's formula: $\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$, we get an output of the form:

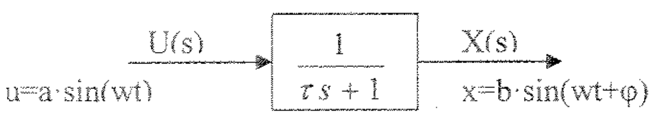
$$x(t) = a\,|G(j\omega)|\sin(\omega t + \phi)$$

Summary:
If we input a sine wave of amplitude a to a stable linear system, the output will be a sine wave of the same frequency with different amplitude and phase.



The output amplitude b can be computed as:
$b = a \cdot |G(jw)|$
The output phase can be computed as:
$\varphi = \angle G(jw)$

Example:



The output amplitude b can be computed as:
$b = a \cdot |G(jw)| = a\frac{1}{\sqrt{1+(\tau\omega)^2}}$

The output phase can be computed as:
$\varphi = \angle G(jw) = 0 - \tan^{-1}\left(\frac{\tau\omega}{1}\right)$